Name (Last, First): ______________________________

Student ID: ______________________________

Circle your section:

201	Shin	8am	71 Evans	212	Lim	1pm	3105 Etcheverry
202	Cho	8am	75 Evans	213	Tanzer	2pm	35 Evans
203	Shin	9am	105 Latimer	214	Moody	2pm	81 Evans
204	Cho	9am	254 Sutardja Dai	215	Tanzer	3pm	206 Wheeler
205	Zhou	10am	254 Sutardja Dai	216	Moody	3pm	61 Evans
206	Theerakarn	10am	179 Stanley	217	Lim	8am	310 Hearst
207	Theerakarn	11am	179 Stanley	218	Moody	5pm	71 Evans
208	Zhou	11am	254 Sutardja Dai	219	Lee	5pm	3111 Etcheverry
209	Wong	12pm	3 Evans	220	Williams	12pm	289 Cory
210	Tabrizian	12pm	9 Evans	221	Williams	3pm	140 Barrows
211	Wong	1pm	254 Sutardja Dai	222	Williams	2pm	220 Wheeler

If none of the above, please explain: ______________________________

This is a closed book exam, no notes allowed. It consists of 8 problems, each worth 10 points. We will grade all 8 problems, and count your top 6 scores.

Problem	Maximum Score	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total Possible	60	

Problem 1) True or False. Decide if each of the following statements is TRUE or FALSE. You do not need to justify your answers. Write the full word **TRUE** or **FALSE** in the answer box of the chart. (Each correct answer receives 2 points, incorrect answers or blank answers receive 0 points.)

Statement	1	2	3	4	5
Answer	False	True	False	True	False

1) If $y(t)$ satisfies $y'(t) = 5y(t)$, then $y(0) = 1$.

We must have $y(t) = ce^{5t}$ so $y(0) = c$ but c can be any number for example 0.

2) If $y(t)$ satisfies $y'(t) = 5y(t)$, then $y(t)$ satisfies $y''(t) - 4y'(t) - 5y(t) = 0$.

The auxiliary equation of $y''(t) - 4y'(t) - 5y(t) = 0$ is $r^2 - 4r - 5 = (r-5)(r+1) = 0$ so $y(t) = ce^{5t}$ is a solution.

3) If an $n \times n$ constant matrix A is not diagonalizable, then the solution set of $\mathbf{y}'(t) = A\mathbf{y}(t)$ has dimension less than n.

There is always a basis of solutions of $\mathbf{y}'(t) = A\mathbf{y}(t)$ though there may not be a basis of eigenvectors of A.

4) The initial value problem $ty''(t) + e^t y'(t) + \sin(t)y(t) = \cos(t), y(1) = 1, y'(1) = -1$ has a unique solution on the domain $(0, \infty)$.

On the domain $(0, \infty)$, the function t is never zero, so we can divide by it and invoke the theorem that the initial value problem has a unique solution.

5) Two functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are linearly dependent if the Wronskian

$$W[f_1, f_2](t) = \det \begin{bmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{bmatrix}$$

is zero at some point of $\mathbb{R}$.

This is true if the functions are solutions to a second order ODE. But otherwise it is not necessarily true for example $f_1(t) = 1$, $f_2(t) = t^2$ are linearly independent but their Wronskian

$$W[f_1, f_2](t) = \det \begin{bmatrix} 1 & t^2 \\ 0 & 2t \end{bmatrix} = 2t$$

is zero at $t = 0$.

Problem 2) Multiple Choice. Determine the correct answer to each of the following questions. You do not need to justify your answers. Write the appropriate letter in the answer box of the chart. (Each correct answer receives 2 points, incorrect answers or blank answers receive 0 points.)

Question	1	2	3	4	5
Answer	A	B	E	B	E

1) What is the reduced row echelon form of the matrix $\begin{bmatrix} 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 \end{bmatrix}$?

$$A)\ \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad B)\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad C)\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$D)\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad E)\ \text{none of the preceding.}$$

2) What is the determinant of the matrix $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 5 & 0 & 0 & 4 \end{bmatrix}$?

$A)\ -120 \quad B)\ -24 \quad C)\ 0 \quad D)\ 24 \quad E)\ 120$

3) What is the first row of the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$?

$A)\ \begin{bmatrix} 0 & -2 & 1 \end{bmatrix} \quad B)\ \begin{bmatrix} 0 & -3 & -6 \end{bmatrix} \quad C)\ \begin{bmatrix} 0 & 6 & -3 \end{bmatrix} \quad D)\ \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \quad E)$ the inverse does not exist.

4) For what pair of real numbers (c_1, c_2) is the matrix $\begin{bmatrix} 2 & 0 \\ c_1 & c_2 \end{bmatrix}$ diagonalizable?

$A)\ (1,2) \quad B)\ (2,-2) \quad C)\ (-2,2) \quad D)\ (-1,2) \quad E)$ none of the preceding.

5) For what triple of real numbers (a_1, a_2, a_3), does the function

$$\langle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rangle = a_1 x_1 y_3 + a_2 x_2 y_2 + a_3 x_3 y_1$$

define an inner product on $\mathbb{R}^3$?

$A)\ (1,0,-1) \quad B)\ (1,-1,1) \quad C)\ (-1,2,1) \quad D)\ (1,2,1) \quad E)$ none of the preceding.

Problem 3) Let V be the vector space of differentiable real-valued functions on the interval $[-1, 1]$ with inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$$

Let W be the subspace of V spanned by $1, t^2$.

1) (5 points) Find an orthogonal basis of W.

Apply Gram-Schmidt to find orthogonal second basis element

$$t^2 - \frac{\langle 1, t^2 \rangle}{\langle 1, 1 \rangle} 1 = t^2 - \frac{1}{3}$$

2) (5 points) Find the the function in the line spanned by t closest to e^t.

The function will be given by the orthogonal projection

$$\frac{\langle e^t, t \rangle}{\langle t, t \rangle} t = 3e^{-1}t$$

We calculated the numerator by integration by parts. Using $\frac{d}{dt}(e^t t) = e^t t + e^t$, we have $\int_{-1}^{1} e^t t dt = e^t t|_{-1}^{1} - \int_{-1}^{1} e^t dt = (e + e^{-1}) - (e - e^{-1}) = 2e^{-1}$.

Problem 4) 1) (5 points) Find the general solution of the third order ODE

$$y''' - 2y'' + 2y' = 0$$

The auxiliary equations is $r^3 - 2r^2 + 2r = r(r^2 - 2r + 2) = 0$. Its roots are $r = 0, (2 \pm \sqrt{4-8})/2 = 1 \pm i$. Thus the general solution is $c_1 + c_2 e^{(1+i)t} + c_3 e^{(1-i)t}$. If one prefers real-valued functions, the general solution can be alternatively expressed as $d_1 + d_2 e^t \cos(t) + d_3 e^t \sin(t)$.

2) (5 points) Find the general solution of the third order ODE

$$y''' - 2y'' + 2y' = t$$

Applying the method of undetermined coefficients, we try $y = a_0 + a_1 t + a_2 t^2$ and calculate $y''' - 2y'' + 2y' = -4a_2 + 2a_1 + 4a_2 t$. So we can take $a_0 = 0, a_2 = 1/4$ and hence $a_1 = 1/2$. The general solution is then

$$y = t/2 + t^2/4 + d_1 + d_2 e^t \cos(t) + d_3 e^t \sin(t)$$

Problem 5) Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

1) (5 points) Find the general solution of the equation $\mathbf{y}'(t) = A\mathbf{y}(t)$.

Eigenvalues are $\lambda = 1, 2+i, 2-i$ with eigenvectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix}$$

So general solution is

$$\mathbf{y}(t) = c_1 e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{(2+i)t} \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} + c_3 e^{(2-i)t} \begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix}$$

or alternatively

$$\mathbf{y}(t) = d_1 e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + d_2 e^{2t} \begin{bmatrix} \cos(t) \\ 0 \\ -\sin(t) \end{bmatrix} + d_3 e^{2t} \begin{bmatrix} \sin(t) \\ 0 \\ \cos(t) \end{bmatrix}$$

2) (5 points) Find a solution of the equation $\mathbf{y}'(t) = A\mathbf{y}(t)$ such that $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

From the above, we have

$$\mathbf{y}(0) = d_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So we take $d_1 = 2, d_2 = 1, d_3 = 3$ and so

$$\mathbf{y}(t) = 2e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + e^{2t} \begin{bmatrix} \cos(t) \\ 0 \\ -\sin(t) \end{bmatrix} + 3e^{2t} \begin{bmatrix} \sin(t) \\ 0 \\ \cos(t) \end{bmatrix}$$

Problem 6) Consider the heat equation on the rod $[0, \pi]$ with temperature zero boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0.$$

1) (5 points) Find the initial temperature if the temperature at time $t = 1$ is given by $-e^{-1}\sin(x)$.

Recall the solutions to the heat equation with $\beta = 1, L = \pi$ given by $u_n(x,t) = e^{-n^2 t}\sin(nx)$. At $t = 1$ we have $u_n(x,1) = e^{-n^2}\sin(nx)$ so in particular for $n = 1$ we have $u_1(x,1) = e^{-1}\sin(x)$. Thus the sought after solution is given by $u(x,t) = -e^{-t}\sin(x)$ and at $t = 0$ we find $u(x,0) = -\sin(x)$.

2) (5 points) Prove that if the temperature at $t = 2014$ is zero throughout the rod, then the initial temperature was zero throughout the rod.

Recall we can write the general solution as a series $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin(nx)$. At $t = 2014$ we have $u(x, 2014) = \sum_{n=1}^{\infty} c_n e^{-2014n^2}\sin(nx)$. If this equals zero, then the Fourier coefficients $c_n e^{-2014n^2}$ must all be equal to zero since the functions $\sin(nx)$, $n = 1, 2, 3, \ldots$ are orthogonal and hence linearly independent. Thus the original coefficients c_n must all be equal to zero and hence $u(x,t)$ is equal to zero.

Problem 7) (10 points) Consider the cosine Fourier series

$$|\sin(x)| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

for any real number x.

Find a_0, a_1, a_2, a_3. (It may be useful to recall $\sin(u)\cos(v) = \frac{1}{2}(\sin(u+v) + \sin(u-v))$.)

For any $n \geq 0$, we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos nx \; dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin x \cos nx \; dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} (\sin(1+n)x + \sin(1-n)x) \; dx \end{aligned} \tag{1}$$

and we consider two cases $n = 1$ and $n \neq 1$. If $n = 1$, then (1) gives

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin 2x \; dx = \frac{1}{2\pi}(-\cos 2x)|_0^{\pi} = 0$$

and if $n \neq 1$ then (1) gives

$$\begin{aligned} a_n &= \frac{1}{\pi}\left(\frac{-1}{1+n}\cos(1+n)x + \frac{-1}{1-n}\cos(1-n)x\right)|_0^{\pi} \\ &= \frac{-1}{\pi}\left(\frac{1}{1+n} + \frac{1}{1-n}\right)((-1)^{n+1} - 1) \\ &= \frac{2}{\pi(1-n^2)}((-1)^n + 1) \end{aligned}$$

thus $\boxed{a_0 = \frac{4}{\pi}, a_1 = 0, a_2 = -\frac{4}{3\pi}, a_3 = 0}$.

Problem 8) (10 points) Decide if the following assertion is TRUE or FALSE. If true, provide a proof; if false provide a counterexample.

Assertion: If $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ and $\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ solve the equation

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{y}(t)$$

with $\mathbf{u}(0) \perp \mathbf{v}(0)$ and $\mathbf{u}(1) \perp \mathbf{v}(1)$, then $\mathbf{u}(t) \perp \mathbf{v}(t)$ for all t.

The assertion is TRUE. Here is a proof.

A basis of solutions is given by

$$\mathbf{y}_1(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{y}_2(t) = e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Suppose $\mathbf{u}(t) = a_1\mathbf{y}_1(t) + a_2\mathbf{y}_2(t)$, $\mathbf{v}(t) = b_1\mathbf{y}_1(t) + b_2\mathbf{y}_2(t)$. Then we need

$$\langle \mathbf{u}(t), \mathbf{v}(t) \rangle = 2a_1b_1e^{2t} + 2a_2b_2e^{-2t}$$

to be zero for all t.

The conditions $\mathbf{u}(0) \perp \mathbf{v}(0)$, $\mathbf{u}(1) \perp \mathbf{v}(1)$ imply

$$2a_1b_1 + 2a_2b_2 = 0 \quad 2a_1b_1e^2 + a_2b_2e^{-2} = 0$$

So $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and also $\begin{bmatrix} e^2b_1 \\ e^{-2}b_2 \end{bmatrix}$.

Case 1: If $b_1 = b_2 = 0$, then the assertion is clearly true.

Case 2: If $b_1 = 0, b_2 \neq 0$, then $a_2 = 0$ and the assertion is clearly true.

Case 3: If $b_1 \neq 0, b_2 = 0$, then $a_1 = 0$ and the assertion is clearly true.

Case 4: If $b_1 \neq 0, b_2 \neq 0$, then

$$\det \begin{bmatrix} b_1 & e^2b_1 \\ b_2 & e^{-2}b_2 \end{bmatrix} = b_1b_2(e^{-2} - e^2) \neq 0$$

so the vectors $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and $\begin{bmatrix} e^2b_1 \\ e^{-2}b_2 \end{bmatrix}$ are linearly independent. Thus $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ must be zero since it is orthogonal to both. The assertion is now clearly true.